

The background features a repeating geometric pattern. It consists of a grid of white lines forming a series of interlocking shapes. The primary shapes are triangles and hexagons, each containing intricate, stylized designs in red and blue. Interspersed among these are smaller, golden, star-like or floral motifs. The overall effect is a dense, colorful, and highly symmetrical tessellation.

Math 3500

# Algebra I: Group Theory

Book 2

Similar to HW#2: How many elements of each order does  $S_4$  have?

1 element of order 1:  $() = \text{identity}$   
 9 elements of order 2:  $(12), (13), (14), (23), (24), (34),$   $\leftarrow \binom{4}{2} = 6$  transpositions  
 $(12)(34), (13)(24), (14)(23)$   $\leftarrow \frac{1}{2!} \binom{4}{2} \binom{2}{2} = \frac{6}{2} = 3$   
 8 elements of order 3:  $(123), (124), (132), (134), (142), (143), (234), (243)$   
 6 elements of order 4:  $(1234), (1243), (1324), (1342), (1423), (1432)$   


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 $24 = 4! = |S_4|$        $|S_n| = n! = 1 \times 2 \times 3 \times \dots \times n$

In  $S_n$  the number of  $n$ -cycles is  $(n-1)!$   
 The binomial coefficient  $\binom{n}{k}$  ("n choose k") is the number of ways to choose a subset of size  $k$  from a set of size  $n$ .

$\binom{n}{k} = k^{\text{th}}$  entry in  $n^{\text{th}}$  row of Pascal's Triangle

$\binom{4}{2} = \text{number of 2-subsets in } [4] = \{1, 2, 3, 4\}$   
 $= 6$

By the way,  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$   
 Binomial Theorem  $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

Pascal's Triangle

$n=0$	1							
$n=1$	1	1						
$n=2$	1	2	1					
$n=3$	1	3	3	1				
$n=4$	1	4	6	4	1			
$n=5$	1	5	10	10	5	1		
$n=6$	1	6	15	20	15	6	1	
$n=7$	1	7	21	35	35	21	7	1

A transposition is a 2-cycle  $(i\ j) \in S_n$ ,  $i \neq j$  in  $[n] = \{1, 2, \dots, n\}$ .

Products of disjoint transpositions eg.  $(1\ 3)(2\ 5)(6\ 8) \in S_8$   
are elements of order 2.

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How many elements of order 2 are there in  $S_7$ ?

Transpositions:  $(12), (13), (14), \dots, (67)$  i.e.  $(i\ j)$  where  $i \neq j$  in  $[7] = \{1, 2, \dots, 7\}$

$$\binom{7}{2} = 21 \text{ transpositions}$$

Products of two disjoint transpositions eg.  $(26)(34) = (34)(26)$

$$\text{Number of these is } \frac{1}{2} \binom{7}{2} \binom{5}{2} = 105$$

Products of three disjoint transpositions eg.  $(15)(27)(36) = (15)(36)(27) = (27)(36)(15)$

$$\text{Number of these is } \frac{1}{6} \binom{7}{2} \binom{5}{2} \binom{3}{2} = \frac{21 \times 10 \times 3}{6 \cdot 2} = 105$$

Number of 3-cycles in  $S_7$  eg.  $(274)$ :  $2 \binom{7}{3} = 2 \times 35 = 70$

Number of products of two disjoint 3-cycles: eg.  $(274)(356) = (356)(274)$

$$\frac{1}{2} \cdot 70 \cdot \binom{4}{3} \cdot 2 = 70 \cdot 4 = 280$$

Elements of order 12 in  $S_7$  eg.  $(142)(3756)$

$$70 \cdot 3! = 70 \cdot 6 = 420 \text{ elements of order 12 in } S_7$$

$$280 + 70$$

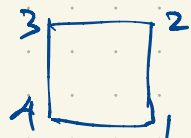
$$= 350$$

elements  
of order 3  
in  $S_7$

Revisiting the dihedral group of order 8 (symmetry group of a square)

$$G = \{ I, R, R^2, R^3, D, D', V, H \}$$

viewed as a group of permutations of the four vertices



$$I \mapsto ()$$

$$R \mapsto (1234)$$

$$R^2 \mapsto (13)(24)$$

$$R^3 \mapsto (1432)$$

$$H \mapsto (12)(34)$$

$$V \mapsto (14)(23)$$

$$D \mapsto (13)$$

$$D' \mapsto (24)$$

$G \cong$  subgroup of  $S_4$ :  $\{(), (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$

$$RV = (1234)(14)(23) = (24) = D' = \langle (1234), (13) \rangle$$

This is an example of a permutation group of degree 4, i.e. a subgroup of  $S_4$ .  
A permutation group of degree  $n$  is a subgroup of the symmetric group  $S_n$ .

Theorem (Cayley's Representation Theorem) Every finite group is isomorphic to a permutation group. In fact, if  $|G| = n$  then  $G$  is isomorphic to a subgroup of  $S_n$ . (But we can usually do better. Eg. the dihedral group of order 8 is isomorphic to a subgroup of  $S_8$ . But  $S_4$  is even better.)

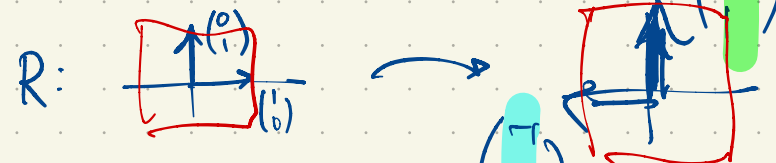
The two most important general classes of examples of groups are  
(i) permutation groups (i.e. subgroups of  $S_n$ ) and  
(ii) linear groups (i.e. subgroups of  $GL_n(F) = \{ \text{invertible } n \times n \text{ matrices over a field } F \}$ ). eg.  $F = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$  (integers mod  $p$ )

The dihedral group of order 8 is also a subgroup of  $GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

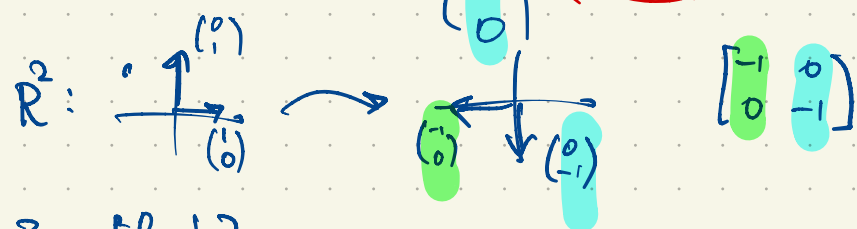
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



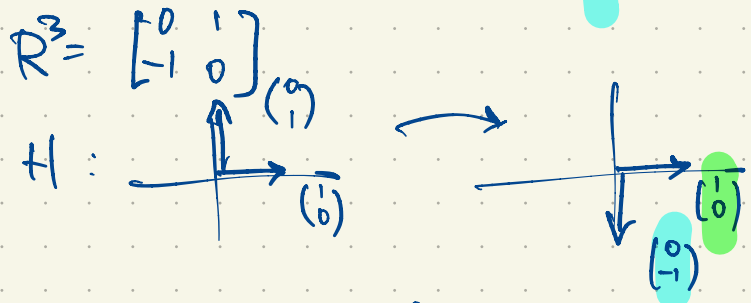
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

(the vector  $\begin{bmatrix} x \\ y \end{bmatrix}$  rotated  $90^\circ$  counter-clockwise about the origin)

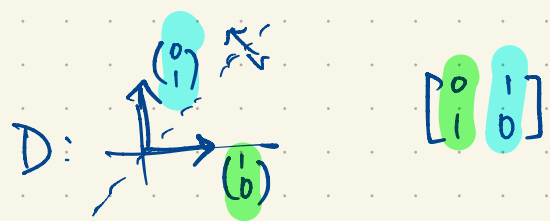


$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

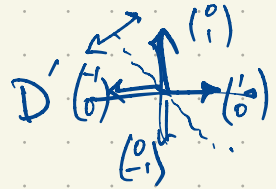


$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$G =$  symmetry group of square  $\cong \left\{ \begin{matrix} I & R & R^2 & R^3 & H \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ V & D & D' \end{matrix} \right\}$



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



subgroup of  $GL_2(\mathbb{R})$

$$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

commutes with every element of  $G$   
(not so immediately obvious from the other ways of representing  $G$ ).

$$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$$

Similar to HW2 #4, 5:  $\mathbb{F}_p = \{0, 1, \dots, p-1\}$  integers mod  $p$ .

eg.  $p=3$ ,  $F = \mathbb{F}_3 = \{0, 1, 2\}$   $F^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in F \right\}$

$$\begin{array}{r|rrr} + & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array}$$

$$\begin{array}{r|rrr} \times & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{array}$$

$$\frac{1}{2} = 2 \quad -\frac{1}{2} = -2 = 1$$

$$|F^2| = 3^2 = 9$$

$2 \times 2$  matrices over  $F$  there are  $3^4 = 81$   $2 \times 2$  matrices over  $F$ .  $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$

$$GL_2(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in F, \underbrace{ad - bc \neq 0} \right\}$$

ie.  $ad - bc = 1$  or  $2$  ie.  $ad - bc = \pm 1$ .

$$|GL_2(F)| = 8 \times 6 = 48$$

$9-1=8$  choices for first column to be nonzero  
 $9-3=6$  choices for the second column to be not a scalar multiple of the first column.

eg.  $g = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \in GL_2(F) \Rightarrow g^2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

so  $|g| = 2$ .

Eg.  $F = \mathbb{F}_2 = \{0, 1\}$

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \quad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

ie.  $\det A = 1$

Let's try  $G = GL_3(F) = \{3 \times 3 \text{ matrices } A \text{ over } F \text{ such that } \det A \neq 0\}$

The total number of  $3 \times 3$  matrices over  $F$  is  $2^9 = 512$ .

$$|G| = 7 \times 6 \times 4 = 168$$

8-1 nonzero vectors  
8-2  
8-4

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$F^3$  is a 3-dimensional vector space over  $F$

$$|F^3| = 2^3 = 8$$

eg.  $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in G$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^3 = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}_{A^2} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$|A| = 4 = \text{the order of } A$   
Note:  $4 \mid 168$

$$A^4 = A^2 A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$F^3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\begin{matrix} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{matrix}$

$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  permutes the vectors as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

ie. (4217)  
= (1742)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{ie. } (36)$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{ie. } (5)$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{ie. } (0)$$

$$= (1742)(36)$$

has order 4

$B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  what is the order of  $B \in GL_3(F)$ ,  $F = \{0, 1\}$  ?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$v_4 \quad v_2 \quad v_1 \quad v_6 \quad v_3 \quad v_7 \quad v_5 \quad v_4$

ie.  $B = (1637542)$  has order 7.

Alternatively compute  $B, B^2, B^3, B^4, B^5, B^6, B^7 = I$ .

$$|S_7| = 7! = 5040$$

$\langle A, B \rangle \cong \langle (1742)(36), (1637542) \rangle$  subgroup of  $S_7$  of order 168  
 $\cong GL_3(F)$

Note: If  $F = \mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$  (integers mod  $p$ ,  $p$  prime)

and  $G = GL_3(F) =$  group of invertible  $3 \times 3$  matrices over  $F$

then  $|G| = (p^3 - 1)(p^3 - p)(p^3 - p^2) = p^3(p^3 - 1)(p^2 - 1)(p - 1)$

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

↑  
nonzero vectors

↑  
not scalar multiple of first col.

$p^9$   $3 \times 3$  matrices in all;  
but not all of them are invertible.

$$\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$$

$$\mathbb{F}_7^* = \{1, 2, 3, 4, 5, 6\}$$

multiplicative group  
of order 6

$$\begin{aligned} \langle 1 \rangle &= \{1\} & \langle 6 \rangle &= \{1, 6\} \\ \langle 2 \rangle &= \{1, 2, 4\} \\ \langle 3 \rangle &= \{1, 3, 2, 6, 4, 5\} = \langle 5 \rangle \\ \langle 4 \rangle &= \{1, 4, 2\} = \langle 2 \rangle \\ \langle 5 \rangle &= \{1, 5, 4, 6, 2, 3\} \end{aligned}$$

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

=

x	1	3	2	6	4	5
1	1	3	2	6	4	5
3	3	2	6	4	5	1
2	2	6	4	5	1	3
6	6	4	5	1	3	2
4	4	5	1	3	2	6
5	5	1	3	2	6	4

$$\mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\} \text{ under addition mod 6}$$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

We have an isomorphism from  
 $(\mathbb{Z}/6\mathbb{Z}, +)$  to  $(\mathbb{F}_7^*, \cdot)$

defined by  $\phi: \mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{F}_7^*$

$$\phi(0) = 1$$

$$\phi(1) = 3$$

$$\phi(2) = 2$$

$$\phi(3) = 6$$

$$\phi(4) = 4$$

$$\phi(5) = 5$$

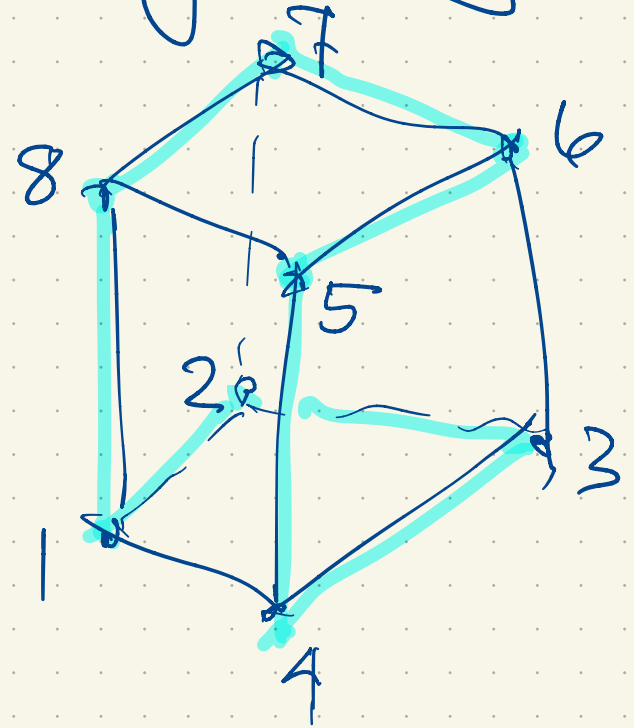
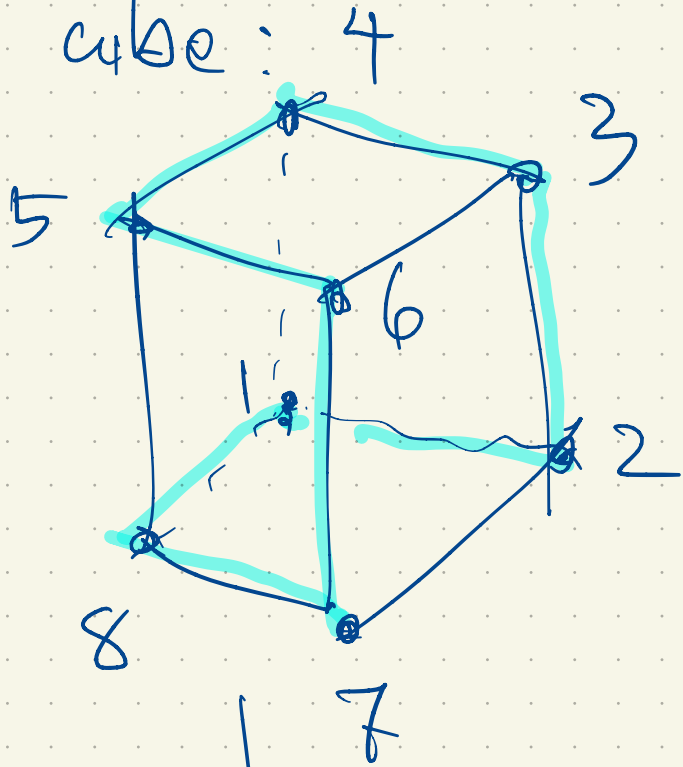
$$\phi(x+y) = \phi(x)\phi(y)$$

$$\phi(\underbrace{1+1+\dots+1}_k \text{ times}) = \underbrace{3 \cdot 3 \cdot \dots \cdot 3}_k \text{ times}$$

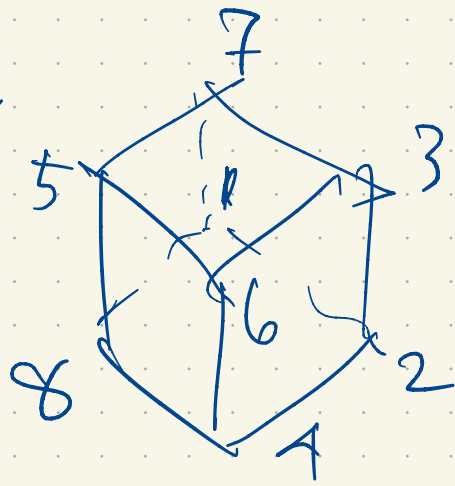
$$\phi(k) = 3^k$$

$$\begin{aligned} \phi(k+l) &= 3^{k+l} = 3^k \cdot 3^l \\ &= \phi(k)\phi(l) \end{aligned}$$

check that  $(123658)(47)$  is a symmetry of  
my cube:



$(47)$  is not a symmetry



March 2026 - Apr 2026

M	W	F
9	11	13
16	18	20
23	25	27
30	1	3

Test on Mon Mar 30  
covering material prior to  
today

The symmetric group  $S_n = \{\text{permutations of } [n]\}$   
where  $n = \{1, 2, \dots, n\}$  Note: there are  
 $n$  points being permuted by.  $|S_n| = n! = 1 \times 2 \times \dots \times n$   
permutations. THIS IS NOT THAT.

There are  $\binom{n}{2} = \frac{n(n-1)}{2}$  transpositions in  $S_n$ .

Theorem  $S_n$  is generated by its transpositions

eg. for  $n=4$ ,  $S_4 = \langle (12), (13), (14), (23), (24), (34) \rangle$ .

eg.  $(13)(12) = (123) = (14)(23)(12)(34)(24)(12)$

Theorem Every permutation  $\sigma \in S_n$   
is either even or odd, never both.  
( $\sigma$  is even if it's a product of an  
even number of transpositions;  
 $\sigma$  is odd if it's a product of  
an odd number of transpositions.)

$\sum$  upper case Sigma M  
 $\sigma$  lower case sigma  $\mu$  mu  
or  $\sigma$

k	$\sigma(k)$
1	2
2	3
3	1
4	4
5	5

$\sigma = (123)$

To prove the theorem, we must show is that  
if  $\sigma = \tau_1 \tau_2 \tau_3 \dots \tau_{2k+1}$  then  $\sigma = \mu_1 \mu_2 \dots \mu_{2l}$

$\tau_i$  transpositions

$\mu_j$  transpositions

$\mu \mu$

$$() = (12)(12)$$

Proof Consider the polynomial

$$f(x_1, x_2, x_3, x_4) = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3)$$

$$f(x_1, x_2, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i) \quad \text{of degree } \binom{n}{2}.$$

$$\text{If } \sigma \in S_n \text{ then } f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) = \pm f(x_1, x_2, x_3, x_4)$$

eg. for  $(12)$  we get

$$\begin{aligned} f(x_2, x_1, x_3, x_4) &= (x_1 - x_2)(x_3 - x_2)(x_4 - x_2)(x_3 - x_1)(x_4 - x_1)(x_4 - x_3) \\ &= - f(x_1, x_2, x_3, x_4) \end{aligned}$$

for  $(123)$  we get

$$\begin{aligned} f(x_2, x_3, x_1, x_4) &= (x_3 - x_2)(x_1 - x_2)(x_4 - x_2)(x_1 - x_3)(x_4 - x_3)(x_4 - x_1) \\ &= f(x_1, x_2, x_3, x_4) \end{aligned}$$

The sign of  $\sigma \in S_n$  has value  $\text{sgn}(\sigma) = \pm 1$  where  $f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = (\text{sgn}(\sigma)) f(x_1, x_2, \dots, x_n)$

$$\text{For } \sigma, \sigma' \in S_n, \quad \text{sgn}(\sigma\sigma') = \text{sgn}(\sigma) \text{sgn}(\sigma').$$

$\underbrace{\quad}_{\pm 1} \quad \underbrace{\quad}_{\pm 1} \quad \underbrace{\quad}_{\pm 1}$

i.e.

even	x	even	=	even
even	x	odd	=	odd
odd	x	even	=	odd
odd	x	odd	=	even

where  $\sigma$  is even if  $\text{sgn}(\sigma) = 1$ ;  
odd if  $\text{sgn}(\sigma) = -1$ .

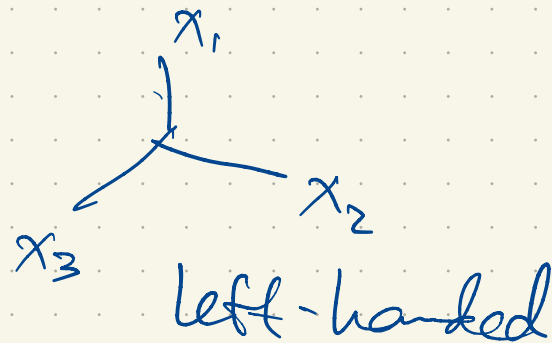
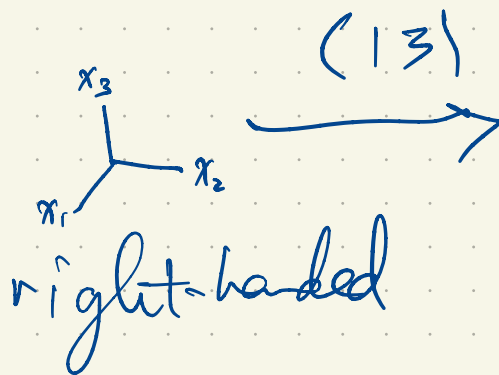
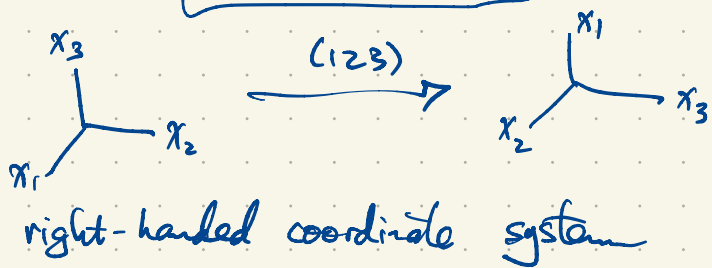
Note: Cycles of even length are odd permutations.  
Cycles of odd length are even permutations.

An  $n$ -cycle (i.e. a cycle of length  $n$ ) is a product of  $n-1$  transpositions.

eg.  $(12)$  is an odd permutation  
 $(13)(12) = (123)$  is an even permutation  
 $(14)(13)(12) = (1234)$  is an odd permutation.

$$\underbrace{(1\ 5\ 7\ 6\ 4)}_{\text{even}} \underbrace{(2\ 3\ 8\ 9)}_{\text{odd}} \quad \text{is odd}$$

$$= \underbrace{(14)(16)(17)(15)}_{\text{even}} \underbrace{(29)(28)(23)}_{\text{odd}}$$



$(123)$  preserves orientation  
 $(13)$  reverses orientation

An  $n \times n$  matrix  $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$  has determinant

$$\det A = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1, \sigma(1)} a_{2, \sigma(2)} \dots a_{n, \sigma(n)}$$

( $n!$  terms)

If  $n=3$  then

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33} - a_{11} a_{23} a_{32}$$

1	2	3	4
5	6	7	8
9	10	12	14
13	15	11	



1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	



$(11, 15, 14, 12)$

$(11, 12, 14, 15)$

is an ~~even~~ <sup>odd</sup> permutation