

Putnam Team Seminar

Practice Problems 9

Monday, November 4, 2024

1. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a non-zero function g defined on (a, b) such that this wrong product rule is true for x in (a, b) .

2. In the additive group of ordered pairs of integers (m, n) [with addition defined componentwise: $(m, n) + (m', n') = (m+m', n+n')$] consider the subgroup H generated by the three elements $(3, 8)$, $(4, -1)$, $(5, 4)$. Then H has another set of generators of the form $(1, b)$, $(0, a)$ for some integers a, b with $a > 0$. Find a .

[Elements g_1, g_2, \dots, g_k are said to *generate* a subgroup H if (i) each $g_i \in H$, and (ii) every $h \in H$ can be written as a sum $h = n_1g_1 + n_2g_2 + \dots + n_kg_k$ where the n_i are integers (and where, for example, $3g_1 - 2g_2$ means $g_1 + g_1 + g_1 - g_2 - g_2$).]

3. Evaluate $\sum_{k=0}^n (-1)^k \binom{n}{k} (x - k)^n$.

4. For which real numbers c is there a straight line that intersects the graph of $f(x) = x^4 + 9x^3 + cx^2 + 9x + 4$ in four distinct points?

5. Show that there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

6. Prove that for each positive integer n , the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

7. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?