Practice Problems 8

Monday, October 28, 2024

1. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

2. Show that the improper integral $\displaystyle\lim_{B\to\infty}\int_0^B \sin(x)\sin(x^2)\,dx$ converges.

3. Evaluate $\displaystyle\int_0^\infty \Bigl(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\Bigr)\Bigl(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\Bigr)\,dx.$

4. Let G be a group with identity e and $\phi : G \to G$ a function such that
$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$
whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element a in G such that $\psi(x) = a\phi(x)$ is homomorphism (that is, $\psi(xy) = \psi(x)\psi(y)$ for all x and y in G).

5. Let a convex polygon P be contained in a square of side one. Show that the sum of the squares of the sides of P is less than or equal to 4.

6. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

7. Let k and n be integers with $1 \leqslant k \leqslant n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?