Practice Problems 6

Monday, October 7, 2024

1. If z is any complex root of $11z^{10} + 10iz^9 + 10iz - 11 = 0$, show that $|z| = 1$. (Here i is a complex number satisfying $i^2 = -1$.)

2. Let n be a positive integer, and define $f(n) = 1! + 2! + \cdots + n!$. Find polynomials $P(x)$ and $Q(x)$ such that
$$f(n{+}2) = P(n)f(n{+}1) + Q(n)f(n)$$
for all $n \geqslant 1$.

3. Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of th numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}$, $\{4, 8\}$, $\{5, 7\}$ shows that the largest k is *at least* 3.]

4. Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that for all x,
$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1, & \text{if } x < -1; \\ 3x + 2, & \text{if } -1 \leqslant x \leqslant 0; \\ -2x + 2, & \text{if } x > 0. \end{cases}$$

5. Given a set of six points in the plane, prove that the ratio of the longest distance between any pair, to the shortest, is at least $\sqrt{3}$.

6. If a linear transformation A on an n-dimensional vector space has $n{+}1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

7. Determine all solutions in real numbers x, y, z, w of the system
$$x + y + z = w, \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}.$$