Practice Problems 5

Monday, September 30, 2024

1. In the (x, y)-plane, if R is the set of points inside and on a convex polygon, let $D(x, y)$ be the distance from (x, y) to the nearest point of R.
   (a) Show that there exist constants a, b, c, independent of R, such that
   $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-D(x,y)}\,dx\,dy = a + bL + cA,$$
   where L is the perimeter of R and A is the area of R.
   (b) Find the values of a, b, and c.

2. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$ where a, b, c, d are integers.

3. Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S, the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

4. Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

5. Let n be a positive integer. How many solutions are there in the ordered positive integer pairs (x, y) to the equation
   $$\frac{xy}{x+y} = n?$$

6. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that
   $$f'(x) = \frac{f(x+n) - f(x)}{n}$$
   for all real numbers x and all positive integers n.

7. Let $p(x)$ be a polynomial that is non-negative for all x. Prove that, for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that
   $$p(x) = \sum_{j=1}^{k} f_j(x)^2.$$