

Putnam Team Seminar

Practice Problems 4

Monday, September 23, 2024

1. The graph of the equation $x^y = y^x$ in the first quadrant (i.e., the region where $x > 0$ and $y > 0$) consists of a straight line and a curve. Find the coordinates of the intersection point of the line and the curve.
2. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.
3. Given any set of ten consecutive positive integers, there exists an integer in the list that is pairwise relatively prime with each one of the other integers in the list.

4. Evaluate $\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$.

5. Let $n = 2m$, where m is an odd integer greater than 1. Let $\theta = e^{2\pi i/n}$. Express $(1 - \theta)^{-1}$ as a polynomial in θ ,

$$a_k \theta^k + a_{k-1} \theta^{k-1} + \cdots + a_1 \theta + a_0,$$

with *integer coefficients* a_i . (Note that θ is a primitive n -th root of unity, and thus it satisfies all of the identities that hold for such roots.)

6. Let A and B be matrices of size 3×2 and 2×3 respectively. Suppose that

$$AB = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}.$$

Show that

$$BA = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

7. The function $f(x)$ has a continuous derivative satisfying $0 < f'(x) \leq 1$ for $0 \leq x \leq 1$. Also suppose $f(0) = 0$. Show that

$$\left[\int_0^1 f(x) dx \right]^2 \geq \int_0^1 f(x)^3 dx.$$

(*Hint:* You might consider rewriting integrals in terms of the inverse function.) Also give an example in which equality holds.