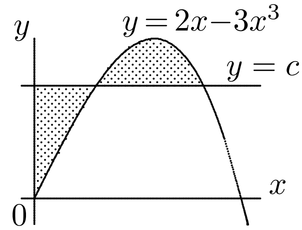
Practice Problems 3

Monday, September 16, 2024

1. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



2. A game of solitaire is played as follows. After each play, according to the outcome, the player receives a or b points (a and b are positive integers with a greater than b), and his score accumulates from play to play. It has been noticed that there are thirty-five non-attainable scores and that one of them is 58. Find a and b.

3. A **composite** (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

4. Let $F(x)$ be a real valued function defined for all real x except for $x = 0$ and $x = 1$ and satisfying the functional equation $F(x) + F(\frac{x-1}{x}) = 1 + x$. Find all functions $F(x)$ satisfying these conditions.

5. Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation
$$h(x, y) \;=\; a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$
for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \leqslant M$ for all (x, y) in $\mathbb{R}^2$, then h is identically zero.

6. Consider the power series expansion
$$\frac{1}{1 - 2x - x^2} \;=\; \sum_{n=0}^{\infty} a_n x^n.$$
Prove that, for each integer $n \geqslant 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$.

7. Show that the closed unit disk in the plane cannot be partitioned into two disjoint congruent subsets.