Practice Problems 11

Monday, November 18, 2024

1. Show that $\displaystyle\int_0^1 x^x dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}$.

2. Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

3. Let $f(x)$ be a continous function such that $f(2x^2-1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leqslant x \leqslant 1$.

4. Let $u(t)$ be a continuous function in the system of differential equations
$$\frac{dx}{dt} = -2y + u(t), \qquad \frac{dy}{dt} = -2x + u(t).$$
Show that, regardless of the choice of $u(t)$, the solution of the system which satisfies $x = x_0$, $y = y_0$ at $t = 0$ will never pass through $(0,0)$ unless $x_0 = y_0$. When $x_0 = y_0$, show that, for any positive value t_0 of t, it is possible to choose $u(t)$ so the solution is at $(0,0)$ when $t = t_0$.

5. Find the minimum value of
$$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$$
for $0 < u < \sqrt{2}$ and $v > 0$.

6. Suppose that a, b, c, A, B, C are real numbers, $a \neq 0$ and $A \neq 0$, such that
$$\left|ax^2 + bx + c\right| \leqslant \left|Ax^2 + Bx + C\right|$$
for all real numbers x. Show that
$$\left|b^2 - 4ac\right| \leqslant \left|B^2 - 4AC\right|.$$

7. Let G be a group, with operation $*$. Suppose that
   (i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);
   (ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

   Prove that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for all $\mathbf{a}, \mathbf{b} \in G$.