University of Wyoming

Math 2850—Fall 2024

Department of Mathematics

Putnam Team Seminar

Practice Problems 10

Monday, November 11, 2024

1. P is an interior point of the angle whose sides are the rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Locate X on $\overrightarrow{OA}$ and Y on $\overrightarrow{OB}$ so tht the line segment XY contains P and so that the product of the distances $(PX)(PY)$ is a minimum.

2. Evaluate $\displaystyle\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx\,.$

3. Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers.

   [*Example*: $0 = 0^2+0^2$, $1 = 0^1+1^2$, and $2 = 1^2+1^2$.]

4. Let $f(x,y)$ be a polynomial with real coefficients in the real variables x and y defined over the entire x-y plane. What are the possibilities for the range of $f(x,y)$?

5. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

6. Let $a_1 < a_2 < a_3 < \cdots$ be an increasing sequence of positive integers. Let the series
$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$
   be convergent. For any number x, let $k(x)$ be the number of the a_n's which do not exceed x. Show that $\lim\limits_{n\to\infty} k(x)/x = 0$.

7. Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove $B = 1$.