University of Wyoming

Math 2850—Fall 2024

Department of Mathematics

Putnam Team Seminar

Practice Problems 1

Monday, August 26, 2024

1. Find positive integers n and $a_1, a_2, \ldots, a_n$ such that
$$a_1 + a_2 + \cdots + a_n = 1991$$
and the product $a_1 a_2 \cdots a_n$ is as large as possible.

2. For any square matrix A, we can define $\sin A$ by the usual power series:
$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$
Prove or disprove: There exists a 2×2 matrix A with real entries such that
$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

3. Evaluate
$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

4. Let S be a set and '$\star$' a binary operation on S satisfying
$$x \star (x \star y) = y \qquad \text{and} \qquad (y \star x) \star x = y$$
for every x and y in S. Show that $\star$ is commutative but not necessarily associative.

5. Show that if f is real-valued and continuous on $(-\infty, \infty)$, and $\int_{-\infty}^{\infty} f(x)dx$ exists, then
$$\int_{-\infty}^{\infty} f\left(x - \tfrac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

6. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

7. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$, the polynomial
$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
has exactly n distinct real roots?